Fixed Point Theorem for Four Mappings in Cone Metric Spaces Using EA and CLR Property

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Abstract:
In this paper, we present new fixed point theorems for four mappings A,B,S,T defined on a complete cone metric space by employing the (EA) **property and the** Common Limit Range (CLR) **property. These concepts allow the establishment of fixed point results under weaker conditions than those used in classical fixed point theory. We extend and generalize existing results by introducing contractive conditions involving the CLR property, without requiring continuity or compactness. Examples are provided to demonstrate the applicability of our results.**

Keyword: Cone metric space, common fixed point, EA, CLR.

1. INTRODUCTION:

Pant [9,10] was among the first to explore common fixed points of non-compatible mappings in metric spaces. Later, Aamri and El Moutawakil [5] introduced the (E.A) property, which generalizes the concept of non-compatibility, and established several common fixed point theorems under strict contractive conditions.
Huang and Zhang [4] extended the classical framework by introducing cone metric spaces, a generalization of metric spaces. They also developed the notion of convergence in such spaces and proved fixed point theorems for contractive mappings. Following their work, many researchers [4,6,7,8] contributed by establishing fixed point theorems for self-mappings in cone metric spaces. Additionally, the author in [11] presented common fixed point results for pairs of non-self-mappings defined on closed subsets of metrically convex cone metric spaces.
Further developments include studies on quasi-contractions in cone metric spaces [13,16], where fixed point theorems were derived under various conditions. In [1], the results of [2] were extended to pairs of self-mappings. The authors in [15] established common fixed point theorems under strict contractive conditions, while in [3], fixed point results were obtained for set-valued mappings.
In this paper, we investigate common fixed points of four self-mappings A,B,S,T defined on a cone metric space. we extend previous results by removing the necessity for continuity or compatibility. This generalization is useful in the study of coupled systems and operator equations.

2. PRELIMINARIES:

Definition 2.1: Let E be a real Banach Spaces. A subset P of E is called a cone if and only if

a. P is closed ,non empty and $p \neq 0$
b. $a , b \in$ R $, a , b \geq 0$ and $x, y \in P$ imply $ax + by \in P$
c. $P \cap (-P) = \{0\}$

Given a cone $P \subset E$ we define the partial ordering ≤ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote that $x \leq y$ but $x \neq y$, while $x << y$ will stand for

$$y - x \in int. P .$$

Definition 2.2: Let X be a nonempty set. Suppose the mapping $d: X \times X \rightarrow E$ satiafies the following condition:

(i). $0 < d(x,y) \forall x,y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$
(ii). $d(x,y) = d(y,x)$, $\forall x,y \in X$
(iii). $d(x,y) \leq d(x,y) + d(x,y)$, $\forall x,y \in X$
Then d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2.3: Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X, $\{x_n\}$ is a Cauchy sequence if there is some $k \in \mathbb{N}$ such that, for all $n,m \geq k$,
$d(x_n, x_m) \ll c$;

Definition 2.4: Let (X,d) be a conemetric space, $\{x_n\}$ a sequence in X, $\{x_n\}$ is a convergent sequence if there is some $k \in \mathbb{N}$ such that, for all $n \geq k$,

$$d(x_n, x) \ll c;$$

Then x is called limit of the sequence $\{x_n\}$
Note that:- (i) Every convergent sequence in a cone metric space X is a Cauchy sequence.
(ii) A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 2.5: Let (S,T) be a pair of self-mappings of a cone metric space (X,d). Then the pair (S,T) is said to be non-compatible if there exists at least one sequence in X such that

$$\lim_{n\to\infty} d(Tx_n, x) = \lim_{n\to\infty} d(Sx_n, x) = 0$$

for some $x \in X$ but

$$\lim_{n\to\infty} d(STx_n, TSx_n) \text{ is either non} - \text{zero or non} - \text{existent}$$

Definition 2.6:[5]Let S and T be self mapping of a metric space (X,d). We say that S and T satisfy E.A. property if there exist a sequences $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

for some $t \in X$.
The class of E.A. mappings contains the class of non-compatible mappings.
Wutiphol Sintunavarat and Poom Kumam (2011)[14] prove some common fixed point theorems under weakly compatible mappings using new property that is CLR (Common Limit Range) property.

Definition 2.7:[14] (CLR Property) Suppose that (X,d) is a metric space and $S, T : X \to X$. If $S$and T satisfies following condition:

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sx \text{ for some } x \in X.$$

Then the mappings are said to satisfy the common limit in the range of S property.
In what follows, the common limit in the range of S property will be denoted by the (CLRs) property.

3. MAIN RESULTS

Theorem 3.1:- Let F, G, H, and L be mappings on cone metric space (X,d) into itself with satisfying the conditions

i. $$d\big(L(y), H(x)\big) \leq a \max\left\{\frac{1}{2} d\big(L(y), F(x)\big), d\big(H(x), F(x)\big)\right\}$$

$$+ b d\big(G(y), H(x)\big) + c \frac{d\big(H(x), F(x)\big)}{1 + d\big(L(y), F(x)\big)} \qquad \ldots(3.1)$$

For all $x, y \in X$, $a, b, c \geq 0$ and $a + b + c < 1$
ii. $H(X) \subseteq G(X)$ and $L(X) \subseteq F(X)$
iii. The pairs (H,F) and (L,G) are weakly compatible.
iv. One of pair satisfies property E. A.
Then F , G, H and L have a unique common fixed point.
Proof:- Let (L,G) satisfy E.A. property then by definition there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathrm{L}\{x_n\} = \lim_{n\to\infty} \mathrm{G}\{x_n\} = t \qquad \text{for some } t \in X \qquad \ldots(3.2)$$

Since $L(X) \subseteq F(X)$ there exist a sequence $\{y_n\}$ in X such that $L(x_n) = F(y_n)$

Hence $\lim_{n\to\infty} \mathrm{F}\{y_n\} = t$, we claim that $\lim_{n\to\infty} \mathrm{H}\{y_n\} = t$ if not, we putting $x = y_n$ and $y = x_n$ in (3.1)

$$d\big(L(x_n), H(y_n)\big) \le a \max\left\{\frac{1}{2} d\big(L(x_n), F(y_n)\big), d\big(H(y_n), F(y_n)\big)\right\}$$

$$+bd\big(G(x_n), H(y_n)\big) + c\frac{d\big(H(y_n), F(y_n)\big)}{1 + d\big(L(x_n), F(y_n)\big)}$$

From above condition we get

$$d\big(L(x_n), H(y_n)\big) \le a \max\left\{\frac{1}{2} d\big(L(x_n), L(x_n)\big), d\big(H(y_n), L(x_n)\big)\right\}$$

$$+bd\big(G(x_n), H(y_n)\big) + c\frac{d\big(H(y_n), L(x_n)\big)}{1 + d\big(L(x_n), L(x_n)\big)}$$

$$(1 - a - c)d\big(L(x_n), H(y_n)\big) \le bd\big(G(x_n), H(y_n)\big)$$

Letting $n \to \infty$ we have

$$(1 - a - c)d\big(t, H(y_n)\big) \le bd\big(t, H(y_n)\big)$$

$$(1 - a - b - c)d\big(t, H(y_n)\big) \le 0$$

which is contradiction.

Hence $\lim_{n\to\infty} \mathrm{H}\{y_n\} = \lim_{n\to\infty} \mathrm{F}\{y_n\} = t$

Now suppose first $F(X)$ is complete subspace of X then $t = F(u)$ for some $u \in X$ then

$$\lim_{n\to\infty} \mathrm{L}\{x_n\} = \lim_{n\to\infty} \mathrm{G}\{x_n\} = \lim_{n\to\infty} \mathrm{H}\{y_n\} = \lim_{n\to\infty} \mathrm{F}\{y_n\} = t = F(u)$$

We claim that $H(u) = F(u)$ if not, we putting $x = u$ and $y = x_n$ in (3.1)

$$d\big(L(x_n), H(u)\big) \le a \max\left\{\frac{1}{2} d\big(L(x_n), F(u)\big), d\big(H(u), F(u)\big)\right\}$$

$$+bd\big(G(x_n), H(u)\big) + c\frac{d\big(H(u), F(u)\big)}{1 + d\big(L(x_n), F(u)\big)}$$

From above conditions and letting $n \to \infty$ we have

$$d\big(t, H(u)\big) \le a \max\left\{\frac{1}{2} d(t, t), d(H(u), t)\right\} + bd\big(t, H(u)\big) + c\frac{d(H(u), t)}{1 + d(t, t)}$$

$$(1 - a - b - c)\, d(t, H(u)) \le 0$$

which is contradiction.

$$H(u) = t$$

Clearly $H(u) = F(u) = t$ Hence u is coincidence point of (H, F).

Now the weak compatibility (H, F) implies that $HF(u) = FH(u)$ or $Ht = Ft$

On the other hand $H(X) \subseteq G(X)$ there exist $v \in X$ such that $H(u) = G(v)$.

Thus, $H(u) = F(u) = G(v) = t$

Let us show that v is coincidence point of (L,G) that is $G(v) = L(v) = t$ if not then putting $x = u$ and $y = v$ in (3.1) we have

$$d\big(L(v), H(u)\big) \le a \max\left\{\frac{1}{2} d\big(L(v), F(u)\big), d\big(H(u), F(u)\big)\right\}$$

$$+bd\big(G(v), H(u)\big) + c\frac{d\big(H(u), F(u)\big)}{1 + d\big(L(v), F(u)\big)}$$

From above conditions we get

$$d\big(L(v),H(u)\big) \le a\max\left\{\frac{1}{2}d\big(L(v),H(u)\big),d\big(H(u),H(u)\big)\right\}$$

$$+bd\big(G(v),H(u)\big)+c\frac{d\big(H(u),H(u)\big)}{1+d\big(L(v),H(u)\big)}$$

Letting $n\to\infty$ we have

$d(L(v),t)\le a\max\left\{\frac{1}{2}d(L(v),t),d(t,t)\right\}+bd(t,t)$

$d(L(v),t)\le a\frac{1}{2}d(L(v),t)$

$\left\{1-\frac{a}{2}\right\}d(L(v),t)\le 0$

which is contradiction.

$L(v)=t$

Clearly, $L(v)=G(v)=t$ Hence v is coincidence point of (L,G).

Now the weak compatibility of pair (L,G) implies that $GL(u)=LG(u)$ or $Lt=Gt$.

Therefore t is common coincidence of F,G,H and L.

In order to show that t is a common fixed point,

let us t not a common fixed point of F,G,H and L,

Let us put $x=u$ and $y=t$ in (3.1) we have,

$$d\big(L(t),H(u)\big)\le a\max\left\{\frac{1}{2}d\big(L(t),F(u)\big),d\big(H(u),F(u)\big)\right\}$$

$$+bd\big(G(t),H(u)\big)+c\frac{d\big(H(u),F(u)\big)}{1+d\big(L(t),F(u)\big)}$$

From above conditions we get

$$d(L(t),t)\le a\max\left\{\frac{1}{2}d(L(t),t),d(t,t)\right\}+bd(G(t),t)+c\frac{d(t,t)}{1+d(L(t),t)}$$

$d(L(t),t)\le a\frac{1}{2}d(L(t),t)+bd(L(t),t)$

$\left(1-\frac{a}{2}-b\right)d(L(t),t)\le 0$

which is contradiction.

$$L(t)=t$$

Clearly $F(t)=H(t)=L(t)=G(t)=t$

Hence t is common fixed of F, G, H and L.

Similar argument arises if we assume that G(X) is complete subspace of X. Similarly the property (E.A.) of the pair (H,F) will give similar result.

For uniqueness of common fixed point, let us assume that w be another common fixed point of F,G,H and L.

Let us put $x=w$ and $y=t$ in (3.1) we have

$$d\big(L(t),H(w)\big)\le a\max\left\{\frac{1}{2}d\big(L(t),F(w)\big),d\big(H(w),F(w)\big)\right\}$$

$$+bd\big(G(t),H(w)\big)+c\frac{d\big(H(w),F(w)\big)}{1+d\big(L(t),F(w)\big)}$$

From above conditions we get

$$d(t,w)\le a\max\left\{\frac{1}{2}d(t,w),d(w,w)\right\}+bd(t,w)+c\frac{d(w,w)}{1+d(t,w)}$$

$d(t,w) \le a\frac{1}{2}d(t,w) + bd(t,w)$
$\left(1-\frac{1}{2}a-b\right)d(t,w) \le 0$
$\Rightarrow d(t,w) = 0 \Rightarrow w = t$
Clearly $F(t) = H(t) = L(t) = G(t) = t$
Hence t is common fixed of F, G, H and L. This completes the proof.

Theorem 3.2:- Let F, G, H, and L be mappings on cone metric space (X, d) into itself with satisfying the conditions

i. $d\big(L(y), H(x)\big) \le a \max\left\{\frac{1}{2}d\big(L(y), F(x)\big), d\big(H(x), F(x)\big)\right\} + b\,\frac{d(H(x),F(x))}{1+d(L(y),F(x))}$ …(3.3)

For all $x, y \in X$, $a, b \ge 0$ and $\frac{a}{2} < 1$, $a + b < 1$

ii. $H(X) \subseteq G(X)$ and $L(X) \subseteq F(X)$

iii. The pairs (H, F) and (L, G) are weakly compatible.

iv. One of pair satisfies property (CLR_L) or (CLR_H) property

Then F, G, H and L have a unique common fixed point.

Proof:- Let (L, G) satisfy (CLR_L) property then by definition there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} Gx_n = Lx \text{ for some } x \in X. \quad \ldots(3.4)$$

Since $L(X) \subseteq F(X)$ we have $L(x) = F(w)$ for some $w \in X$.
We claim that
$Hw = Fw = t$ If not, we putting $x = w$ and $y = x_n$ in (3.3)

$$d\big(L(x_n), H(w)\big) \le a \max\left\{\frac{1}{2}d\big(L(x_n), F(w)\big), d\big(H(w), F(w)\big)\right\} + b\,\frac{d\big(H(w), F(w)\big)}{1 + d\big(L(x_n), F(w)\big)}$$

Letting $n \to \infty$ and using above condition we get

$$d\big(L(x), H(w)\big) \le a \max\left\{\frac{1}{2}d\big(L(x), F(w)\big), d\big(H(w), F(w)\big)\right\} + b\,\frac{d\big(H(w), F(w)\big)}{1 + d\big(L(x), F(w)\big)}$$
$$d\big(L(x), H(w)\big) \le ad\big(H(w), F(w)\big) + bd\big(H(w), F(w)\big)$$
$$d\big(L(x), H(w)\big) \le ad(H(w), L(x)) + bd(H(w), L(x))$$
$$(1-a-b)d\big(L(x), H(w)\big) \le 0$$

which is contradiction.
Hence $Hw = Lx$ implies that $Fw = Hw = Lx = t$.
Hence w is coincidence point of H and F.
Since the pair$(H, F)$is weak compatible

$$\Rightarrow HFw = FHw = Ht = Ft$$

Further since $H(X) \subseteq G(X)$ there exist some $z \in X$ such that

$$H(w) = G(z)$$

We claim that L(z) = t on the contradiction we put $x = w$ and $y = z$.

$$d\big(L(z), H(w)\big) \le a \max\left\{\frac{1}{2}d\big(L(z), F(w)\big), d\big(H(w), F(w)\big)\right\} + b\,\frac{d\big(H(w), F(w)\big)}{1 + d\big(L(z), F(w)\big)}$$
$$d\big(L(z), H(w)\big) \le a \max\left\{\frac{1}{2}d\big(L(z), H(w)\big), d\big(H(w), H(w)\big)\right\} + b\,\frac{d\big(H(w), H(w)\big)}{1 + d\big(L(z), H(w)\big)}$$

$d\big(L(z), H(w)\big) \le a\,\frac{1}{2}d\big(L(z), H(w)\big)$
$\left(1-\frac{a}{2}\right)d\big(L(z), H(w)\big) \le 0$
$\left(1-\frac{a}{2}\right)d\big(L(z), t\big) \le 0$

$d(L(z), t) = 0$

Thus $Lz = t$ hence $Fw = Hw = Lz = Gz = t$ it shows that z is coincidence point of L and G.

Also the weak compatibility of (L, G) implies that

$$LGz = GLz = Lt = Gt$$

We claim that t is common fixed point of F, G, H and L, on the contradiction let us put $x = w$ and $y = t$

$$d\big(L(t), H(w)\big) \leq a \max\left\{\frac{1}{2} d\big(L(t), F(w)\big), d\big(H(w), F(w)\big)\right\} + b \frac{d\big(H(w), F(w)\big)}{1 + d\big(L(t), F(w)\big)}$$

$$d\big(L(t), H(w)\big) \leq a \max\left\{\frac{1}{2} d\big(L(t), H(w)\big), d\big(H(w), H(w)\big)\right\} + b \frac{d\big(H(w), H(w)\big)}{1 + d\big(L(t), H(w)\big)}$$

$$d\big(L(t), H(w)\big) \leq a\ \frac{1}{2} d\big(L(t), H(w)\big)$$

$$\left(1 - \frac{a}{2}\right) d(L(t), t) \leq 0$$

$$\left(1 - \frac{a}{2}\right) d(L(t), t) \leq 0$$

$$d(L(t), t) = 0$$

Thus $Lt = t$ hence, $Ft = Ht = Lt = Gt = t$.

It shows that t is common fixed point of $F, G, H$and L.

Similarly, the argument that the pair (H, F) satisfy property (CLR_H) will also give the unique common fixed point of F, G, H and L. As a result, we arrive at the same conclusion in both cases the existence and uniqueness of the common fixed point of F, G, H and L. This completes the proof.

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