Lyapunov-Based Generalized Function Projective Synchronization of Discrete-Time Chaotic Systems with Different Dimensions

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Abstract:
This paper develops a novel framework for generalized function projective synchronization (GFPS) in discrete-time chaotic systems with mismatched dimensions. By constructing appropriate nonlinear controllers and employing Lyapunov stability theory, sufficient conditions for global asymptotic synchronization are derived in the form of tractable algebraic criteria. Unlike existing approaches that primarily address continuous-time systems or identical-dimensional models, the proposed method extends GFPS to heterogeneous discrete systems. Numerical simulations using 3D Hénon-like and Fold maps validate the theoretical results and demonstrate rapid convergence of synchronization errors. The proposed scheme is simple, robust, and suitable for practical applications in secure communication and nonlinear signal processing.

Keywords: generalized function projective synchronization, discrete-time chaotic systems, Lyapunov stability, chaos synchronization.
2020 Mathematics Subject Classification (MSC2020): 37D45, 37N35, 34H10, 39A28, 37C75, 93C55

1. INTRODUCTION

In recent years, synchronization of chaotic systems has drawn much attention due to its wide applications in various fields of physics and engineering [1-3]. Different methods have been successfully applied to chaos and hyperchaos synchronization such as the PC method, OGY method, active and adaptive control, backstepping design, and sliding mode control.

Most work in chaos synchronization has been concentrated on continuous-time rather than discrete-time chaotic systems. However, in practice, discrete-time chaotic dynamical systems play a more important role than their continuous counterparts. In fact, many mathematical models of physical, biological, and chemical processes are defined using chaotic dynamical systems in discrete time. Therefore, it is important to consider chaos (hyperchaos) synchronization in discrete-time dynamical systems. Recently, more and more attention has been paid to the synchronization of chaos (hyperchaos) in discrete-time dynamical systems due to their applications in secure communication and cryptology [11-13].

Over the past few decades, many synchronization types have been discovered in chaotic systems [14-20]. Among all types of synchronization, generalized synchronization (GS) has been extensively studied. In GS, two chaotic systems are said to be synchronized if there exists a functional relationship between the states of the drive (master) and response (slave) systems. However, most methods presented to achieve GS are related to continuous-time systems. Few methods address discrete-time systems.

However, existing studies on generalized function projective synchronization (GFPS) primarily focus on

continuous-time systems, and limited attention has been given to discrete-time systems of different dimensions. Motivated by this gap, the present work develops constructive schemes to investigate generalized function projective synchronization (GFPS) of different-dimensional discrete-time chaotic systems, based on Lyapunov stability theory. We present new synchronization criteria established in the form of simple algebraic conditions which are very convenient to verify.
The main contributions of this paper are summarized as follows:

- A unified GFPS framework for discrete-time chaotic systems with different dimensions is proposed.
- Novel controller design ensuring global asymptotic stability is developed.
- Lyapunov-based sufficient conditions are derived in algebraic form.
- Theoretical results are validated using numerical simulations on standard chaotic maps.

2. MATHEMATICAL BACKGROUND

2.1 Stability theory for discrete time Dynamical System

Consider the discrete dynamical system

$$x(n+1) = f\big(x(n)\big); x_0 = a$$

and let $x(n) = E$ be an equilibrium point, i.e., $f(E) = E$. The stability of the equilibrium can be analyzed by evaluating the derivative of f at E, denoted by $f'(E)$. The following criteria hold:

- If $|f'(E)| < 1$, then the equilibrium E is locally asymptotically stable.
- If $|f'(E)| > 1$, then the equilibrium E is unstable.
- If $|f'(E)| = 1$, then the stability of the equilibrium cannot be determined solely from the first derivative, and higher-order analysis is required.

2.2 Lyapunov Stability Theory

An equilibrium point, x_e of a dynamical system is said to be *Lyapunov stable* if, for every solution that starts sufficiently close to x_e, the trajectory remains in its neighborhood for all future time. Furthermore, the equilibrium point x_e is said to be *asymptotically stable* if it is Lyapunov stable and, in addition, all solutions originating sufficiently close to x_e converge to x_e as time progresses.
Let us consider an autonomous system with equilibrium point at the origin $X = 0$. If there exists a scalar function $V(X)$, defined in a neighborhood U of the origin, such that $V(X)$ satisfies the conditions of a Lyapunov function, then the equilibrium point $X = 0$ is Lyapunov stable.
We now examine the stability of the zero solution for the system given by

$$\frac{dx}{dt} = -2x, \frac{dy}{dt} = x - y.$$

This system is linear, homogeneous, and characterized by constant coefficients. To analyze its stability, we consider a quadratic Lyapunov function of the form

$$V(X) = V(x,y) = ax^2 + by^2$$

where a and b are positive constants to be determined.
For discrete-time systems, stability is determined using:

$$\Delta V(k) = V(x(k+1)) - V(x(k))$$

3. PROBLEM FORMULATION:

3.1 Generalized function projective synchronization, when dimension of master system is greater than the dimension of slave system

Let us consider the drive chaotic system as:

$$X(k+1) = AX(k) + f_1\big(X(k)\big), \tag{1}$$

where $X(k) = \big(x_1(k), \dots, x_n(k)\big)^T \in R^n$ is the state vector and $f_1 : R^n \to R^n$ is continuously differentiable function which represent the nonlinear part of the drive system and $A \in \mathbb{R}^{m\times m}$
Now consider the response system as:

$Y(k+1) = BY(k) + g_1(Y(k)) + U,$

(2)

where $Y(k) = \left(y_1(k), \dots, y_m(k)\right)^T \in R^m, B \in \mathbb{R}^{m\times m}$ that represents the linear part of the dynamic system, $g_1 : R^m \to R^m$ is the nonlinear part of the response system and $U = (u_i)_{1\le i\le m} \in R^m$ is the vector controller.

Definition: The drive system and response system are said to be generalized synchronized with respect to the vector map C if there exists a controller $U = (u_i)_{1\le i\le m} \in R^m$ and a given map $C : R^n \to R^m$ such that the synchronization error $e(k) = Y(k) - C(X(k))$

(3)

such that $\lim_{k\to+\infty} \|e(k)\| = 0.$

To study the generalized synchronization of the discrete time dynamical systems, we discuss the asymptotic stability for the zero solution of synchronization error system. Now defining the error function as

$e(k) = Y(k) - C\left(X(k)\right),$

$e(k+1) = Y(k+1) - C(X(k+1))$

We take the controller U such that the solutions of the error system $e_i(k)$ tend to $0, i = 1,2, \dots, n$, as k goes to ∞. In this case, the error system between the drive system and response system can be derived as

$e(k+1) = BY(k) + g_1 Y(k) + U - C\left(AX(k) + f_1\left(X(k)\right)\right),$

(4)

$e(k+1) = Be(k) - Pe(k)$

To achieve generalized synchronization between both systems, we can choose the vector controller U as $U = C[AX(k) + f_1 X(k)] - g_1 Y(k) - BCX(k) - Pe(k),$

(5)

where $P \in R^{m\times m}$ is an unknown control matrix to be determined. Error system can be described as: $e(k+1) = (B - P)e(k).$

(6)

Theorem 1: Assume that there exists a positive definite matrix P such that $(A-K)^T P(A-K) - P < 0$. then we select P as control matrix such that $Q = [I - (B - P)^T (B - P)]$ is also positive definite matrix, then the drive system and the response system are globally generalized synchronized with respect to C, under the controller law that is the error system is globally asymptotically stable.

Proof: Constructing the Lyapunov function as

$V(e(k)) = e^T(k)e(k),$

(7)

we obtain $\Delta V\left(e(k)\right) = e^T(k+1)e(k+1) - e^T(k)e(k),$

$$\Delta V\left(e(k)\right) = B - P^T e^T(k)(B-P)e(k) - e^T(k)e(k),$$
$$\Delta V\left(e(k)\right) = e^T(k)[(B-P)^T(B-P) - I]e(k),$$
$$\Delta V\left(e(k)\right) = e^T(k)(-Q)e(k),$$
$$\Delta V\left(e(k)\right) = -e^T(k)Qe(k) < 0 \quad (8)$$

Thus, from the Lyapunov stability theory, we get

$\lim_{k\to+\infty} e_i(k) = 0; (i = 1,2, \dots, m)$

(9)

So, the zero solution of the error system (6) is globally asymptotically stable, and therefore, systems (1) and (2) are globally generalized synchronized.

3.2 Generalized function projective synchronization, when dimension of slave system is greater than dimension of master system

Let us consider the drive chaotic system as:

$$X(k+1) = AX(k) + f_2(X(k)), \tag{10}$$

where $X(k) = (x_1(k),\dots,x_m(k))^T \in R^m$ is the state vector and $f_2 : R^m \to R^m$ is the nonlinear part of the drive system.

Now consider the response system as:

$$Y(k+1) = BY(k) + g_2(Y(k)) + U, \tag{11}$$

where $Y(k) = (y_1(k),\dots,y_n(k))^T \in R^n, B$ is an $n \times n$ matrix that represents the linear part of the dynamic system, $g_2 : R^n \to R^n$ is the nonlinear part of the response system and $U = (u_i)_{1 \le i \le n} \in R^n$ is the vector controller.

Definition : The drive system and response system are said to be generalized synchronized with respect to the vector map C if there exists a controller $U = (u_i)_{1 \le i \le n} \in R^n$ and a given map $C : R^m \to R^n$ such that the synchronization error $e(k) = Y(k) - D(X(k))$ (12)

such that $\lim_{k \to +\infty} \|e(k)\| = 0$.

To study the generalized synchronization of the discrete time dynamical systems, we discuss the asymptotic stability for the zero solution of synchronization error system. Now defining the error function as

$e(k) = Xe(k) - D(Y(k))$,

$e(k+1) = Y(k+1) - C(X(k+1))$

We take the controller U such that the solutions of the error system $e_i(k)$ tend to $0, i = 1,2,\dots,n$, as k goes to ∞. In this case, the error system between the drive system and response system can be derived as

$$e(k+1) = AX(k) + g_2X(k) + U - D\left(BY(k) + f_2(Y(k))\right), \tag{13}$$

$e(k+1) = Ae(k) - Se(k)$

To achieve generalized synchronization between both systems, we can choose the vector controller U as $U = D[BY(k) + f_2Y(k)] - g_2Y(k) - ACX(k) - Se(k)$, (14)

where $S \in R^{n\times n}$ is an unknown control matrix to be determined. Error system can be described as: $e(k+1) = (A - S)e(k)$. (15)

Theorem 2: If we select the control matrix S such that $W = [I - (A - S)^T (A - S)]$ is a positive definite matrix, then the drive system and the response system are globally inverse generalized synchronized with respect to C, under the controller law.

Proof: Constructing the Lyapunov function as

$$V(e(k)) = e^T(k)e(k), \tag{16}$$

we obtain $\Delta V(e(k)) = e^T(k+1)e(k+1) - e^T(k)e(k)$,

$$\Delta V(e(k)) = A - S^T e^T(k)(A - S)e(k) - e^T(k)e(k),$$
$$\Delta V(e(k)) = e^T(k)[(A - S)^T(A - S) - I]e(k),$$
$$\Delta V(e(k)) = e^T(k)(-W)e(k),$$

$$\Delta V\big(e(k)\big) = -e^T(k)We(k) < 0 \tag{17}$$

Thus, from the Lyapunov stability theory, we get

$$\lim_{k\to+\infty} e_i(k) = 0; (i = 1,2, \ldots, n) \tag{18}$$

So, the zero solution of the error system (15) is globally asymptotically stable, and therefore, systems (10) and (11) are globally inverse generalized synchronized.

4 NUMERICAL SIMULATIONS

Example 1: Consider the 3D Henon-like map as the drive system and the Fold system as the response system. The 3D Henon map [21] can be described as:

$$\begin{aligned} x_1(k+1) &= 1 + x_3(k) - d_1 x_2^2(k), \\ x_2(k+1) &= 1 + d_2 x_2(k) - d_1 x_1^2, \\ x_3(k+1) &= d_2 x_1(k) \end{aligned} \tag{19}$$

The system (19) shows chaotic behaviour for the following parameters' values $d_1 = 1.4$ and $d_2 = 0.2$ and it is depicted through the Figure 1.

The Fold system [21] can be described as

$$\begin{aligned} y_1(k+1) &= y_2(k) + a y_1(k) + u_1, \\ y_2(k+1) &= b + y_1^2(k) + u_2. \end{aligned} \tag{20}$$

The system (20) shows chaotic behaviour for the following parameters' values $a = -0.1$ and $b = -1.7$ and it is shown through the Figure 2.

In view of equations (1) and (10), (2) and (11), we get

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & d_1 & 0 \\ d_1 & 0 & 0 \end{bmatrix}, f_1\big(x(k)\big) = \begin{bmatrix} 1 - d_1 x_2^2(k) \\ 1 - d_2 x_1^2(k) \\ 0 \end{bmatrix}, B = \begin{bmatrix} a & 1 \\ 0 & 0 \end{bmatrix}, g_1\big(y(k)\big) = \begin{bmatrix} 0 \\ b + y_1^2(k) \end{bmatrix},$$

$$C = \begin{bmatrix} k_1(n) & 0 & 0 \\ 0 & k_2(n) & k_3(n) \end{bmatrix}, U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, P = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}$$

where $k_1(n), k_2(n), k_3(n)$ are the projective function and it is taken as:

$k_1(n) = 0.01x_1(k)$, $k_2(n) = 0.02x_2(k)$, $k_3(n) = 0.03x_3(k)$.

The error functions are defined as

$e_1(k) = y_1(k) - k_1(n)x_1(k)$,

$e_2(k) = y_2(k) - k_2(n)x_2(k) - k_3(n)x_3(k)$.

The evolution of error function is shown in Figure 3.

Now we calculate the controller for the function projective synchronization between considered system as $U = C[AX(k) + f_1X(k)] - g_1Y(k) - BCX(k) - Pe(k)$.

$$U = \begin{bmatrix} k_1(1 + x_3(k) - d_1 x_2^2(k)) \\ k_2(1 + d_2 x_2(k) - d_1 x_1^2(k) + k_3 d_2 x_1(k)) \end{bmatrix} - \begin{bmatrix} a k_1 x_1 + k_2 x_2 + k_3 x_3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ b + y_1^2(k) \end{bmatrix} - \begin{bmatrix} P_1 e_1(k) \\ P_2 e_2(k) \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} k_1(1 + x_3(k) - d_1 x_2^2(k)) - a k_1 x_1(k) - k_2 x_2(k) - k_3 x_3(k) - P_1 e_1(k) - P_2 e_2(k) \\ k_2(1 + d_2 x_2(k) - d_1 x_1^2(k)) + k_3 d_2 x_1(k) - b - y_1^2(k) - P_3 e_1(k) - P_4 e_2(k) \end{bmatrix}$$

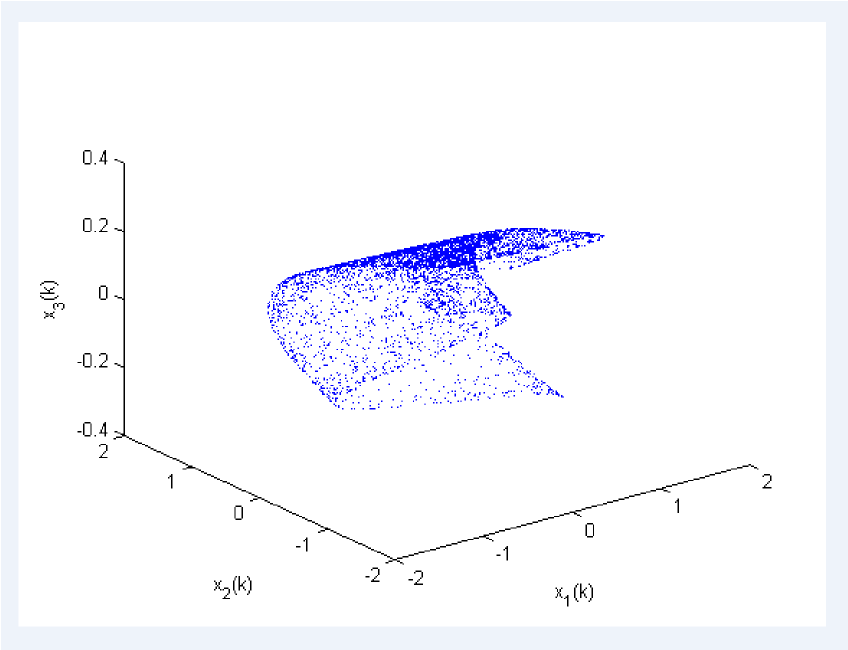


Figure 1: Chaotic attractor of 3D Hénon map (x vs y vs z)

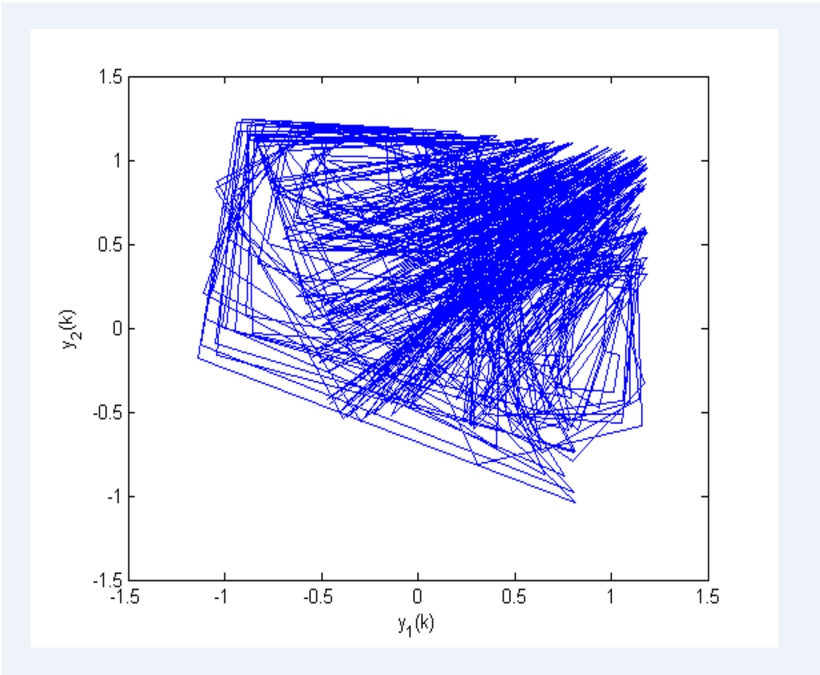


Figure 2: Chaotic attractor of Fold system

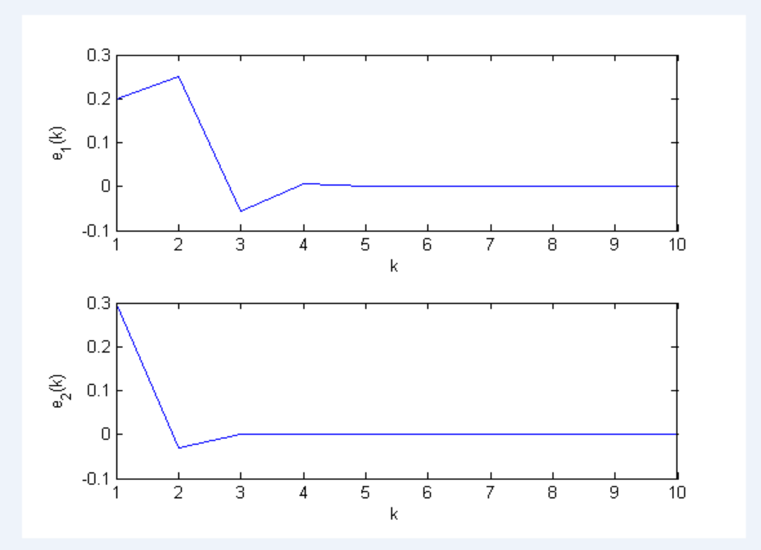


Figure 3: Shows that synchronization errors converge to zero, confirming theoretical results

Example 2: To demonstrate the function projective synchronization, we will consider Fold discrete system as master system and 3D Henon map discrete system as slave system.
The Fold Discrete system [21] is given by

$$x_1(k+1) = x_2(k) + ax_1(k)$$
$$x_2(k+1) = b + x_1(k)^2 \quad (21)$$

The system (21) shows chaotic behavior for the parameters' values: $a = -0.1$ and $b = -1.7$ and it is shown through Figure 5.
The 3 D Henon map [21] is defined as

$$y_1(k+1) = -d_2 y_2(k) + u_1,$$
$$y_2(k+1) = y_3(k) + d_1 y_2(k)^2 + u_2,$$
$$y_3(k+1) = d_2 y_2(k) + y_1(k) + u_3. \quad (22)$$

The system (22) shows chaotic behavior for the parameters' values: $d_1 = 1.07$ and $d_2 = 0.3$ and it is depicted through Figure 4.
The evolution of error function is shown in Figure 6.

In view of equations (1) and (21), (2) and (22), we get

$$A = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}, f_1(X(k)) = \begin{bmatrix} 0 \\ b + x_1(k)^2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & d_2 & 0 \\ 0 & 0 & 1 \\ 1 & d_2 & 0 \end{bmatrix}, g_1(y(k)) = \begin{bmatrix} 0 \\ 1 - y_2(k)^2 \\ 0 \end{bmatrix}, B = \begin{bmatrix} -0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.1 \end{bmatrix}, C = \begin{bmatrix} k_1(n) & 0 \\ 0 & k_2(n) \\ 0 & k_3(n) \end{bmatrix}$$

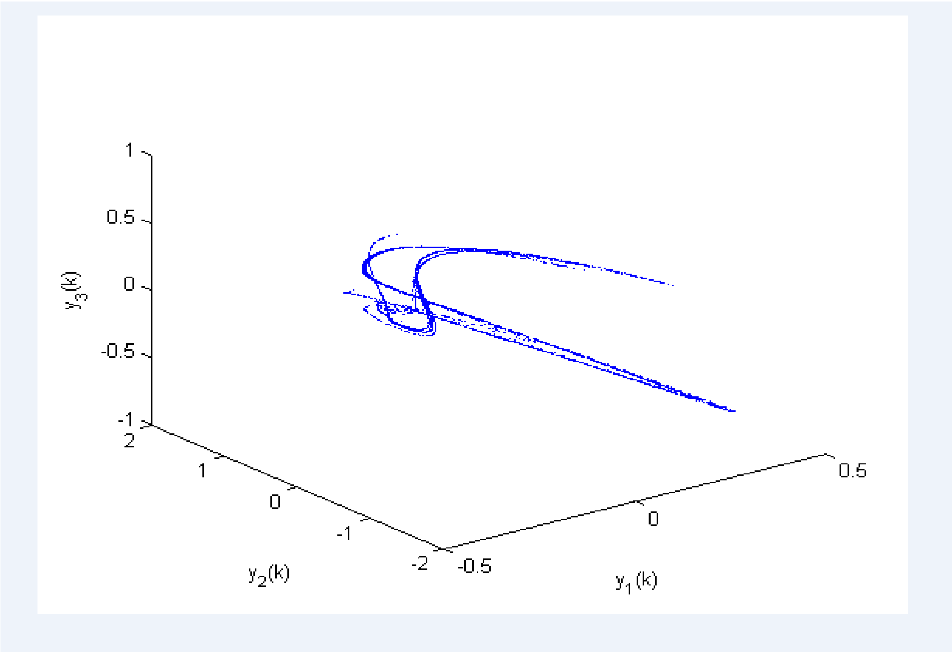
$$\begin{bmatrix} e_1(k) \\ e_2(k) \\ e_3(k) \end{bmatrix} = \begin{bmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix} - \begin{bmatrix} k_1(n) & 0 \\ 0 & k_2(n) \\ 0 & k_3(n) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Where $k_1(n) = 0.02\, x_1(k)$, $k_2(n) = 0.03\, x_2(k)$, $k_3(n) = 0.01\, x_3(k)$ are the function projective and it is considered during the synchronization. The controller for the function projective synchronization can be calculated in view of Theorem 2 as

$$U = \begin{bmatrix} k_1(n) & 0 \\ 0 & k_2(n) \\ 0 & k_3(n) \end{bmatrix} \begin{bmatrix} x_2(k) + ax_1(k) \\ b + x_1(k)^2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 - y_2(k)^2 \\ 0 \end{bmatrix}$$
$$- \begin{bmatrix} 0 & d_2 & 0 \\ 0 & 0 & 1 \\ 1 & d_2 & 0 \end{bmatrix} \begin{bmatrix} k_1(n) & 0 \\ 0 & k_2(n) \\ 0 & k_3(n) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \begin{bmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \\ P_7 & P_8 & P_9 \end{bmatrix} \begin{bmatrix} e_1(k) \\ e_2(k) \\ e_3(k) \end{bmatrix},$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k_1\big(x_2(k) + ax_1(k)\big) + d_2k_2x_2(k) - P_1e_1(k) - P_2e_2(k) - P_3e_3(k) \\ k_2(b + x_1(k)^2) - 1 + d_1y_2(k)^2 - k_3x_2(k) - P_4e_1(k) - P_5e_2(k) - P_6e_3(k) \\ k_3(b + x_1(k)^2) - \mathrm{k}_1x_1(k) - d_2k_3x_2(\mathrm{k}) - P_7e_1(k) - P_8e_2(k) - P_9e_3(k) \end{bmatrix}$$

Figure 4: Chaotic attractor of 3D Henon map



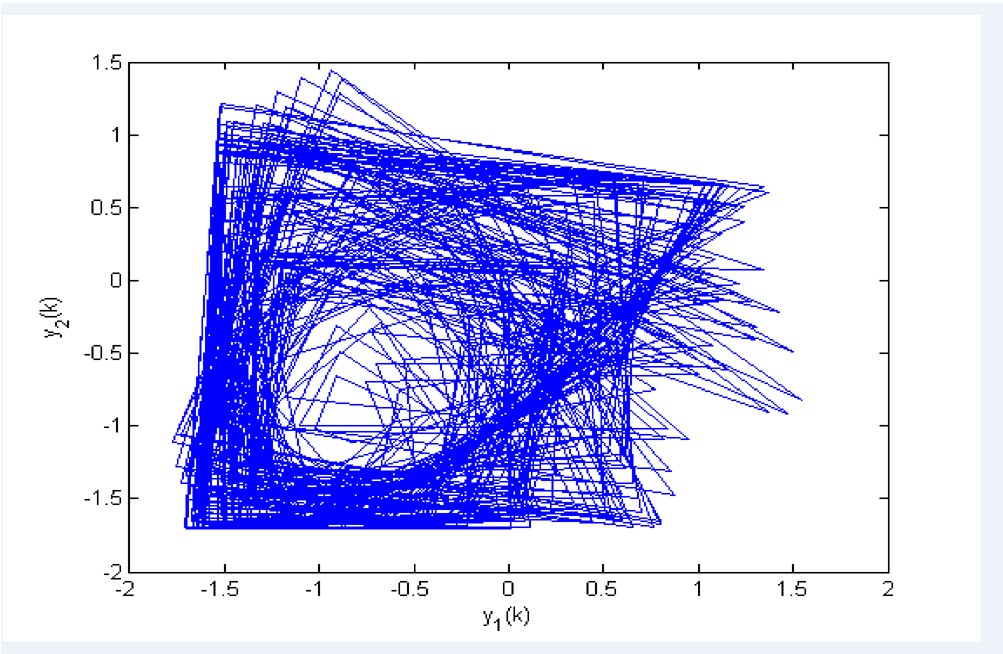


Figure 5: Chaotic attractor of Fold system

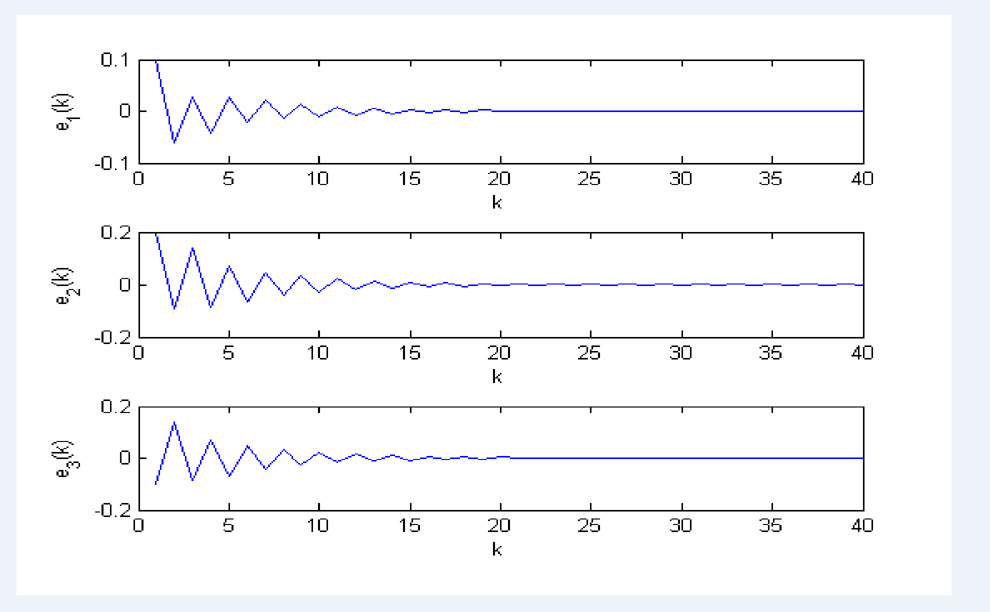


Figure 6: Shows that synchronization errors converge to zero, confirming theoretical results

5 CONCLUSION

In this study, a novel framework for generalized function projective synchronization (GFPS) of discrete-time chaotic dynamical systems with different dimensions has been developed. By employing Lyapunov stability theory, sufficient conditions for achieving synchronization have been established in the form of simple and verifiable algebraic criteria. The proposed control scheme has been designed to effectively stabilize the synchronization error dynamics, thereby ensuring global asymptotic convergence between the drive and response systems.

Unlike many existing approaches that primarily focus on continuous-time systems or systems of identical dimensions, the present work extends the analysis to discrete-time systems with different dimensions, thereby enhancing its applicability to a wider class of practical problems. The introduction of generalized function projective synchronization further provides flexibility in controlling the scaling behaviour between synchronized states, which is of significant importance in applications such as secure communication, signal processing, and complex network analysis.

The effectiveness and validity of the proposed methodology have been demonstrated through numerical simulations, which illustrate the convergence of synchronization errors and confirm the theoretical results. The simplicity of the controller design, combined with the robustness of the Lyapunov-based approach, makes the proposed framework computationally efficient and practically implementable.

In summary, the results presented in this paper contribute to the advancement of synchronization theory in discrete dynamical systems and provide a foundation for further exploration in this direction. Future research may focus on extending the proposed framework to fractional-order systems, stochastic dynamical systems, and large-scale networked systems, as well as investigating robustness under uncertainties and external disturbances. The current work assumes exact system knowledge and does not consider uncertainties or disturbances, which will be addressed in future research.

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